Hausdorff dimension and σ finiteness of $p-$harmonic measures in space when $p \geq n$

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Abstract In this paper we study a measure, $\hat{\mu}$, associated with a positive p harmonic function $\hat{u}$ defined in an open set $O \subset \mathbb{R}^n$ and vanishing on a portion Γ of ∂O. If $p > n$ we show $\hat{\mu}$ is concentrated on a set of σ finite H^{n-1} measure while if $p = n$ the same conclusion holds provided Γ is uniformly fat in the sense of n capacity. Our work nearly answers in the affirmative a conjecture in [14] and also appears to be the natural extension of [10,23] to higher dimensions.

Keywords p harmonic function · p laplacian · p harmonic measure · Hausdorff measure

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1 Introduction

Denote points in Euclidean n-space $\mathbb{R}^n$ by $x = (x_1, \ldots, x_n)$ and let $\bar{E}, \partial E$, diam E, be the closure, boundary, and diameter of the set $E \subset \mathbb{R}^n$. Let $d(E, F)$ be the distance between the sets E, F and $d(y, E) = d(\{y\}, E)$. Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{R}^n$ and let $|x| = \langle x, x \rangle^{1/2}$ be the Euclidean norm of x. Set $B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$ whenever $x \in \mathbb{R}^n$, $r > 0$, and let dx denote Lebesgue n-measure on $\mathbb{R}^n$. If $O \subset \mathbb{R}^n$ is open and $1 \leq q \leq \infty$, then by $W^{1,q}(O)$ we denote the space of equivalence classes of functions f with distributional gradient $\nabla f = (f_{x_1}, \ldots, f_{x_n})$, both of which are q th power integrable on O. Let $\|f\|_{1,q} = \|f\|_q + \| \, |\nabla f| \, \|_q$ be the norm in $W^{1,q}(O)$ where $\| \cdot \|_q$ denotes the usual Lebesgue q norm in O. Next let $C_0^\infty(O)$ be the set of infinitely

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differentiable functions with compact support in O and let $W_0^{1,q}(O)$ be the closure of $C_0^\infty(O)$ in the norm of $W^{1,q}(O)$. If $K \subset \bar{B}(x,r)$ is a compact set let

$$C(K, B(x,2r)) = \inf \int_{\mathbb{R}^n} |\nabla \phi|^n \, dx$$

where the infimum is taken over all $\phi \in W_0^{1,n}(B(x,2r))$ with $\phi \equiv 1$ on K. We say that a compact set $E \subset \mathbb{R}^n$ is locally (n, r_0) uniformly fat or locally uniformly (n, r_0) thick provided there exists $r_0, \beta > 0$, such that whenever $x \in E, 0 < r \leq r_0$,

$$C(E \cap \bar{B}(x,r), B(x,2r)) \geq \beta.$$

Let $O \subset \mathbb{R}^n$ be an open set and $\hat{z} \in \partial O$. Fix $p, 1 < p < \infty$, and suppose that $\hat{u}$ is a positive weak solution to the p Laplace equation in $O \cap B(\hat{z}, \rho)$. That is, $\hat{u} \in W^{1,p}(O \cap B(\hat{z}, \rho))$ and

$$\int |\nabla \hat{u}|^{p-2} \langle \nabla \hat{u}, \nabla \theta \rangle \, dx = 0 \tag{1}$$

whenever $\theta \in W_0^{1,p}(O \cap B(\hat{z}, \rho))$. Equivalently we say that $\hat{u}$ is p harmonic in $O \cap B(\hat{z}, \rho)$. Observe that if $\hat{u}$ is smooth and $\nabla \hat{u} \neq 0$ in $O \cap B(\hat{z}, \rho)$, then $\nabla \cdot (|\nabla \hat{u}|^{p-2} \nabla \hat{u}) \equiv 0$, in the classical sense, where $\nabla \cdot$ denotes divergence. We assume that $\hat{u}$ has zero boundary values on $\partial O \cap B(\hat{z}, \rho)$ in the Sobolev sense. More specifically if $\zeta \in C_0^\infty(B(\hat{z}, \rho))$, then $\hat{u}\,\zeta \in W_0^{1,p}(O \cap B(\hat{z}, \rho))$. Extend $\hat{u}$ to $B(\hat{z}, \rho)$ by putting $\hat{u} \equiv 0$ on $B(\hat{z}, \rho) \setminus O$. Then $\hat{u} \in W^{1,p}(B(\hat{z}, \rho))$ and it follows from (1), as in [9, Chapter 21], that there exists a positive Borel measure $\hat{\mu}$ on $\mathbb{R}^n$ with support contained in $\partial O \cap \bar{B}(\hat{z}, \rho)$ and the property that

$$\int |\nabla \hat{u}|^{p-2} \langle \nabla \hat{u}, \nabla \phi \rangle \, dx = - \int \phi \, d\hat{\mu} \tag{2}$$

whenever $\phi \in C_0^\infty(B(\hat{z}, \rho))$. We note that if ∂O is smooth enough, then $d\hat{\mu} = |\nabla \hat{u}|^{p-1} \, dH^{n-1}$ where H^{n-1} denotes Hausdorff $n-1$ dimensional measure defined after Theorem 1.

In this paper we continue our study of $\hat{\mu}$ for $n \leq p < \infty$. We prove

Theorem 1 *Fix $p, n \leq p < \infty$ and let $\hat{z}, \rho, \hat{u}, \hat{\mu}$ be as in (2). If $p > n$, then $\hat{\mu}$ is concentrated on a set of σ finite H^{n-1} measure. If $p = n$ and $\partial O \cap B(\hat{z}, \rho)$ is locally (n, r_0) uniformly fat, then $\hat{\mu}$ is concentrated on a set of σ finite H^{n-1} measure.*

To define Hausdorff measure and outline previous work we shall need some more notation. If $\lambda > 0$ is a positive function on $(0, \hat{r}_0)$ with $\lim_{r \to 0} \lambda(r) = 0$ define H^λ Hausdorff measure on $\mathbb{R}^n$ as follows: For fixed $0 < \delta < \hat{r}_0$ and $E \subseteq \mathbb{R}^2$, let $L(\delta) = \{B(z_i, r_i)\}$ be such that $E \subseteq \bigcup B(z_i, r_i)$ and $0 < r_i < \delta, \quad i = 1, 2, \ldots$ Set

$$\phi_\delta^\lambda(E) = \inf_{L(\delta)} \sum \lambda(r_i).$$

Then

$$H^\lambda(E) = \lim_{\delta \to 0} \phi_\delta^\lambda(E).$$

In case $\lambda(r) = r^\alpha$ we write H^α for H^λ.

Define the Hausdorff dimension of a Borel measure ν on $\mathbb{R}^n$ by

$$\text{H-dim } \nu = \inf\{\alpha : \exists E \text{ Borel with } H^\alpha(E) = 0 \text{ and } \nu(\mathbb{R}^n \setminus E) = 0\}.$$

From Theorem 1 and the definition of H-dim ν it is easily seen that

Corollary 1 *Let $\hat u, \hat\mu$, be as in Theorem 1. Then H-dim $\hat\mu \le n-1$.*

For $n = 2, 1 < p < \infty$, Lewis proved in [14] the following theorem which generalized earlier results in [4,13,15].

Theorem 2 *Given $p, 1 < p < \infty, p \ne 2$, let $\hat u, \hat\mu$ be as in (1), (2), with $\rho = \infty$ and suppose O is a simply connected bounded domain. Put*

$$\lambda(r) = \lambda(r, A) = r\, \exp[A\sqrt{\log 1/r\, \log\log\log 1/r}], 0 < r < 10^{-6}.$$

Then the following is true.

(a) *If $p > 2$, then $\hat\mu$ is concentrated on a set of σ finite H^1 measure.*
(b) *If $1 < p < 2$, then $\hat\mu$ is absolutely continuous with respect to H^λ provided $A = A(p) \ge 1$ is large enough.*

Remark 1 Makarov in [18] (see also [8,19,21]), essentially proved Theorem 2 for harmonic measure, ω, with respect to a point in O (the $p = 2$ case). Moreover, [10] showed for any planar domain whose complement is a compact set and for which ω exists, that H-dim $\omega \le 1$. Wolff [23] improved this result by showing that for any planar domain ω is concentrated on a set of σ finite H^1 measure.

In higher dimensions, $n \ge 3$, Bourgain [5] showed that H-dim $\omega < n$ for any open set O for which ω exists. Building on an idea of Carleson in [6], Wolff in [24] constructed in $\mathbb{R}^3$, a *Wolff snowflake* for which H-dim $\omega > 2$ and also one for which H-dim $\omega < 2$. This was further generalized in [17] where it was shown that both sides of a Wolff snowflake in $\mathbb{R}^n$ could have harmonic measures, say ω_1, ω_2, with either min(H-dim ω_1, H-dim ω_2) $> n-1$ or max(H-dim ω_1, H-dim ω_2) $< n-1$.

Theorem 4 of [12] implies for fixed p, $1 < p < \infty$, and $\hat u, \hat\mu$ as in (2) that H-dim $\hat\mu < n - \tau$ where $\tau = \tau(p, n) > 0$. Theorem 1 was proved in [16] when $\rho = \infty$ and O is a sufficiently flat Reifenberg domain. Also Wolff's method was extended to the p harmonic setting and produced examples of Wolff type snowflakes and p harmonic functions u_∞ vanishing on the boundary of these snowflakes for which the corresponding measures, say μ_∞, had the following Hausdorff dimensions.

Theorem 3 *If $p \ge n$, then all examples produced by Wolff's method had*

$$\text{H-dim}\ \mu_\infty|_{B(0,1/2)} < n-1.$$

Moreover for $p > 2$, near enough 2, there existed a Wolff snowflake for which

$$\text{H-dim}\ \mu_\infty|_{B(0,1/2)} > n-1.$$

In view of Theorem 3 and the above results it is natural to conjecture that Theorem 1 remains valid for $p = n$ without the uniform fatness assumption on $\partial O \cap B(\hat z, \rho)$. A slightly wilder conjecture is that there exists $p_0, 2 < p_0 < n$, such that if $p_0 \le p$ and $\hat u, \hat\mu$, are the p harmonic function-corresponding measure as in (2), then H-dim $\hat\mu \le n-1$.

As for our proof of Theorem 1, here we first remark that it is embarrassingly simple compared to the proof in Theorem 1(a) of [14]. Moreover the main idea for the proof comes from [23] where a simple proof for harmonic measure in planar domains, whose boundaries are uniformly fat in the sense of logarithmic capacity, is outlined. Our proof also makes important use of work in [14] and [16]. More specifically suppose for fixed $p, 1 < p < \infty$, that $\hat u, \hat\mu, O, \hat z, \rho$ are as in (2). Then from Lemma 4 we see that $\hat u_{x_k}, 1 \le k \le n$, are Hölder continuous in $O \cap B(\hat z, \rho)$. If also $\hat x \in O \cap B(\hat z, \rho)$

and $\nabla \hat{u}(\hat{x}) \neq 0$, then $\hat{u}$ is infinitely differentiable in $B(\hat{x}, \delta)$ for some $\delta > 0$. Let $\xi \in \partial B(0,1)$ differentiating the p Laplace equation, $\nabla \cdot (|\nabla \hat{u}|^{p-2} \nabla \hat{u}) = 0$ with respect to ξ it follows that both $\zeta = \hat{u}_\xi$ and $\zeta = \hat{u}$, satisfy the divergence form PDE for x in $B(\hat{x}, \delta)$:

$$L\zeta(x) \, = \, \sum_{i,k=1}^{n} \frac{\partial}{\partial x_i} \left[\, b_{ik}(x) \zeta_{x_k}(x) \, \right] = 0, \tag{3}$$

where at x

$$b_{ik}(x) = |\nabla \hat{u}|^{p-4} [(p-2) \hat{u}_{x_i} \hat{u}_{x_k} + \delta_{ik} |\nabla \hat{u}|^2](x), \; 1 \leq i, k \leq n, \tag{4}$$

and δ_{ik} is the Kronecker δ. From smoothness of $\hat{u}$ we see that b_{ik} are infinitely differentiable in $B(\hat{x}, \delta)$ and at $x \in B(\hat{x}, \delta)$,

$$\min\{p-1, 1\} |\xi|^2 \, |\nabla \hat{u}(x)|^{p-2} \, \leq \, \sum_{i,k=1}^{n} b_{ik} \, \xi_i \xi_k \, \leq \, \max\{1, p-1\} |\nabla \hat{u}(x)|^{p-2} \, |\xi|^2 \, . \tag{5}$$

The PDE in (3) for $\hat{u}, \hat{u}_{x_k}, 1 \leq k \leq n$, was used in Lemma 5.1 of [15] to show that if $v = \log |\nabla \hat{u}|$ and $\nabla \hat{u}(\hat{x}) \neq 0$, then for $x \in B(\hat{x}, \delta)$,

$$L \, v(x) \geq 0 \text{ when } p \geq n. \tag{6}$$

(3)-(6) are used throughout [4,13,15,16]. Another key inequality in these papers was called the fundamental inequality:

$$\frac{1}{c} \, |\nabla \hat{u}(x)| \leq \, \frac{\hat{u}(x)}{d(x, \partial \Omega)} \, \leq c |\nabla \hat{u}(x)|, \tag{7}$$

where $c = c(n,p)$. (7) was shown to hold for all x near ∂O in the special domains considered in Theorems 2, 3. Observe that if (7) holds, then from (5) it follows that L is locally a uniformly elliptic operator. Hence in these papers results from elliptic PDE were used.

The upper inequality in (7) follows from PDE type estimates and is true for O as in Theorem 1. However the lower estimate is easily seen to fail when ∂O is not connected. Thus we are not able to use either of the strategies in [14] or [15] in our proof of Theorem 1. The argument in section 3 essentially uses only (3) - (6) and the basic estimates for p harmonic functions in section 2.

As for the plan of this paper, in section 2 we list some basic estimates for p harmonic functions. In section 3 we use these estimates and (3)-(6) to prove Theorem 1. Finally in section 4 we make closing remarks and discuss future research.

2 Basic Estimates for p Harmonic Functions.

In the sequel c will denote a positive constant ≥ 1 (not necessarily the same at each occurrence), which may depend only on p, n, unless otherwise stated. In general, $c(a_1, \ldots, a_n)$ denotes a positive constant ≥ 1, which may depend only on $p, n, a_1, \ldots, a_n$ not necessarily the same at each occurrence. $A \approx B$ means that A/B is bounded above and below by positive constants depending only on p, n. In this section, we will always assume that $2 \leq n \leq p < \infty$, and $r > 0$. Let Ω be an open set, $w \in \partial \Omega$, and suppose that $\tilde{u}$ is p harmonic in $\Omega \cap B(w, 4r)$. If $p = n$ we also assume that $\partial \Omega \cap \bar{B}(w, 4r)$ is (n, r_0) uniformly fat as defined above (1).

We begin by stating some interior and boundary estimates for $\tilde{u}$, a positive weak solution to the p Laplacian in $\Omega \cap B(w, 4r)$ with $\tilde{u} \equiv 0$ on $\partial \Omega \cap B(w, 4r)$ in the Sobolev sense, as indicated after (1). Extend $\tilde u$ to $B(w,4r)$ by putting $\tilde u \equiv 0$ on $B(w,4r)\setminus \Omega$. Then there exists a locally finite positive Borel measure $\tilde\mu$ with support $\subset \partial\Omega \cap \bar B(w,4r)$ and for which (2) holds with $\hat u$ replaced by $\tilde u$ and $\phi \in C_0^\infty(B(w,4r))$. Let $\max\limits_{B(z,s)} \tilde u$, $\min\limits_{B(z,s)} \tilde u$ be the essential supremum and infimum of $\tilde u$ on $B(z,s)$ whenever $B(z,s)\subset B(w,4r)$. For proofs of Lemmas 1 - 2 (see [9, Chapters 6 and 7]).

Lemma 1 *Fix $p, 1<p<\infty$, and let $\Omega, w, r, \tilde u$, be as above. Then*

$$\frac{1}{c} r^{p-n} \int\limits_{B(w,r/2)} |\nabla \tilde u|^p \, dx \le \max_{B(w,r)} \tilde u^p \le \frac{c}{r^n} \int\limits_{B(w,2r)} \tilde u^p \, dx.$$

If $B(z,2s)\subset\Omega$, then

$$\max_{B(z,s)} \tilde u \le c \min_{B(z,s)} \tilde u.$$

Lemma 2 *Let $p, \Omega, w, r, \tilde u$, be as in Lemma 1. Then there exists $\alpha=\alpha(p,n)\in(0,1)$ such that $\tilde u$ has a Hölder α continuous representative in $B(w,4r)$ (also denoted $\tilde u$). Moreover if $z_1, z_2 \in B(w,r)$ then*

$$|\tilde u(z_1)-\tilde u(z_2)| \le c \left(\frac{|z_1-z_2|}{r}\right)^\alpha \max_{B(w,2r)} \tilde u$$

Lemma 3 *Let $p, \Omega, w, r, \tilde u$, be as in Lemma 1 and let $\tilde\mu$ be the measure associated with $\tilde u$ as in (2). Then there exists $c, \gamma=\gamma(p,n)\ge 1$, such that*

$$\frac{1}{c}\, r^{p-n}\, \tilde\mu[B(w,r/2)] \le \max_{B(w,r)} \tilde u^{p-1} \le c\, r^{p-n}\, \tilde\mu[B(w,2r)].$$

For the proof of Lemma 3 see [11]. The left-hand side of the above inequality is true for any open Ω and $p\ge n$. However the right-hand side of this inequality requires uniform fatness when $p=n$ and is the main reason we have this assumption in Theorem 1. The reader is referred to [4] for references concerning the proof of the next lemma.

Lemma 4 *Let $p, \Omega, w, r, \tilde u$, be as in Lemma 1. Then $\tilde u$ has a representative in $W^{1,p}(B(w,4r))$ with Hölder continuous partial derivatives in $\Omega\cap B(w,4r)$. In particular, there exists $\sigma\in(0,1]$, depending only on p,n, such that if $x,y\in B(\hat w,\hat r/2)$, $B(\hat w,4\hat r)\subset\Omega\cap B(w,4r)$, then*

$$\frac{1}{c}\,|\nabla\tilde u(x)-\nabla\tilde u(y)| \le \left(\frac{|x-y|}{\hat r}\right)^\sigma \max_{B(\hat w,\hat r)} |\nabla \tilde u| \le \frac{c}{\hat r}\left(\frac{|x-y|}{\hat r}\right)^\sigma \max_{B(\hat w,2\hat r)} \tilde u.$$

If $x\in B(\hat w,4\hat r)$ and $\nabla\tilde u(x)\ne 0$, then $\tilde u$ is infinitely differentiable in an open neighborhood of x. Moreover,

$$\int\limits_{B(\hat w,\hat r)\cap\{|\nabla\tilde u|>0\}} |\nabla\tilde u|^{p-2} \sum_{i,j=1}^n \tilde u_{x_i x_j}^2 \, dx \le \frac{c}{\hat r^2} \int\limits_{B(\hat w,2\hat r)} |\nabla \tilde u|^p\, dx.$$

Lemma 5 *Let $p, \Omega, w, r, \tilde u$, be as in Lemma 1. Suppose for some $z\in\mathbb{R}^n, t\ge 100r$, that $w\in\partial B(z,t)$ and*

$$B(w,4r)\setminus \bar B(z,t) = B(w,4r)\cap\Omega.$$

There exists $\sigma = \sigma(p,n) \in (0,1)$ for which $\tilde u|_{\Omega \cap B(w,3r)}$ has a $C^{1,\sigma} \cap W^{1,p}$ extension to $B(w,3r)$ (denoted $\bar u$). If $x \in B(w,3r) \setminus \partial B(z,t)$ and $\nabla \bar u(x) \neq 0$, then $\bar u$ is infinitely differentiable in an open neighborhood of x. Moreover,

$$\int\limits_{\Omega \cap B(w,r/2) \cap \{|\nabla \bar u|>0\}} |\nabla \bar u|^{p-2} \sum_{i,j=1}^{n} \bar u_{x_i x_j}^2 \, dx \leq \frac{c}{r^2} \int\limits_{\Omega \cap B(w,2r)} |\nabla \bar u|^p \, dx$$

and if $x, y \in \Omega \cap B(w,r/2)$, then

$$\begin{aligned} \tfrac{1}{c} \, |\nabla \bar u(x) - \nabla \bar u(y)| &\leq \left(\tfrac{|x-y|}{r}\right)^{\sigma} \max_{\Omega \cap B(w,r)} |\nabla \bar u| \\ &\leq \tfrac{c}{r} \left(\tfrac{|x-y|}{r}\right)^{\sigma} \max_{\Omega \cap B(w,2r)} \bar u. \end{aligned}$$

Proof We assume as we may that $z = 0$ and $t = 1$ since otherwise we consider $u^*(x) = \tilde u(z + tx)$ and use translation - dilation invariance of the p Laplacian. Let

$$\bar u(x) = \begin{cases} \tilde u(x) & \text{when } x \in \bar\Omega \cap B(w,3r) \\ -\tilde u(\frac{x}{|x|^2}) & \text{when } x \in B(0,1) \cap B(w,3r). \end{cases}$$

If $y = x/|x|^2 \in B(0,1) \cap B(w,3r)$ and $\nabla \tilde u(x) \neq 0$, one can use the chain rule to calculate at y that

$$\nabla \cdot \left(|y|^{2p-2n} |\nabla \bar u|^{p-2} \nabla \bar u\right) = \sum_{i=1}^{n} \frac{\partial}{\partial y_i} \left(|y|^{2p-2n} |\nabla \bar u|^{p-2} \frac{\partial \bar u}{\partial y_i}\right) = 0. \tag{8}$$

Put

$$\gamma(x) = \begin{cases} |x|^{2p-2n} & \text{when } |x| \leq 1 \\ 1 & \text{when } |x| > 1. \end{cases}$$

We assert that $\bar u$ is a weak solution in $B(w,3r)$ to

$$\nabla \cdot \left(\gamma |\nabla \bar u|^{p-2} \nabla \bar u\right) = 0. \tag{9}$$

Indeed from the assumptions on $\tilde u$ we see that $\bar u \in W^{1,p}(B(w,3r))$. Let $\phi \in C_0^\infty(B(w,3r))$ and put

$$\phi_1(x) = \frac{1}{2}(\phi(x) - \phi(\frac{x}{|x|^2}))$$

while

$$\phi_2(x) = \frac{1}{2}(\phi(x) + \phi(\frac{x}{|x|^2})).$$

Using the change of variables theorem and the knowledge garnered from (8) we see that

$$\int\limits_{B(w,4r)} \gamma |\nabla \bar u|^{p-2} \nabla \bar u \cdot \nabla \phi_2 \, dx = 0$$

and

$$\int\limits_{B(w,4r)} \gamma |\nabla \bar u|^{p-2} \nabla \bar u \cdot \nabla \phi_1 dx = 2 \int\limits_{\Omega \cap B(w,4r)} |\nabla \tilde u|^{p-2} \nabla \tilde u \cdot \nabla \phi_1 \, dx = 0$$

Since $\phi = \phi_1 + \phi_2$, we conclude from the above displays that $\bar{u}$ is a weak solution to (9) in $B(w, 3r)$.

From our assertion we see that $\bar{u}$ satisfies the hypotheses in [22], except for γ being continuously differentiable. However the argument in [22] and all constants use only Lipschitzness of γ, so is also valid in our situation. Applying the results in [22] (similar to Lemma 4) and using the definition of $\bar{u}$, we obtain the first and second displays in Lemma 5. □

Lemma 6 *Let $p, \Omega, w, r, \tilde{u}$, be as in Lemma 1 and $-\infty < \eta \leq -1$. Let $L, (b_{ik})$ be as in (3), (4), when $x \in \Omega \cap B(w, 4r)$ and $\nabla \tilde{u}(x) \neq 0$. Let $b_{ij} = \delta_{ij}$ when $\nabla \tilde{u}(x) = 0$ and put $v = \max\{\log |\nabla \tilde{u}|, \eta\}$. Then v is locally a weak sub solution to L in $\Omega \cap B(w, 4r)$.*

Proof From Lemma 4 we see that v is locally in $W^{1,2}(\Omega \cap B(w, 4r))$. Given $\epsilon, \delta, \sigma > 0$ small define g by

$$g(x) = (\max\{v - \eta - \epsilon, 0\} + \sigma)^{\delta} - \sigma^{\delta}, x \in \Omega \cap B(w, 4r).$$

As mentioned earlier in Lemma 5.1 of [15] we showed that $Lv \geq 0$ at $x \in \Omega \cap B(w, 4r)$ when $v(x) \neq \eta$. For the reader's convenience we repeat this calculation after the proof of Lemma 6.

From this fact we deduce that if $0 \leq \theta \in C_0^{\infty}(\Omega \cap B(w, 4r))$, then

$$\begin{aligned} 0 \leq \int_{\Omega \cap B(w,4r)} \theta g L v dx &= -\sum_{i,k=1}^{n} \int_{\Omega \cap B(w,4r)} b_{ik} (\theta g)_{x_i} v_{x_k} dx \\ &\leq -\sum_{i,k=1}^{n} \int_{\Omega \cap B(w,4r)} g b_{ik} \theta_{x_i} v_{x_k} dx, \end{aligned}$$

where in the last inequality we have used (5). Using the above inequality, the bounded convergence theorem, and letting first ϵ, second σ, and third $\delta \to 0$, we get Lemma 6. □

To show $Lv(x) \geq 0$ when $v(x) \neq \eta$, put $\tau(x) = 2v(x) = \log |\nabla \tilde{u}|^2$. We calculate at x,

$$\tau_{x_j} = \sum_{k=1}^{n} \frac{2\tilde{u}_{x_k} \tilde{u}_{x_k x_j}}{|\nabla \tilde{u}|^2}.$$

Furthermore,

$$\begin{aligned} L\tau &= \sum_{i,j,k=1}^{n} \left(b_{ij} \frac{2\tilde{u}_{x_k} \tilde{u}_{x_k x_j}}{|\nabla \tilde{u}|^2} \right)_{x_i} \\ &= \sum_{i,j,k=1}^{n} \frac{2\tilde{u}_{x_k}}{|\nabla \tilde{u}|^2} \left(b_{ij} \tilde{u}_{x_k x_j} \right)_{x_i} + \sum_{i,j,k=1}^{n} 2 b_{ij} \, \tilde{u}_{x_k x_j} \left(\frac{\tilde{u}_{x_k}}{|\nabla \tilde{u}|^2} \right)_{x_i}. \end{aligned}$$

The first term on the right is zero since $L\tilde{u}_{x_k} = 0$ (see (3)). We differentiate the second term to get

$$L\tau = \sum_{i,j,k=1}^{n} \left[2\, |\nabla \tilde{u}|^{-2} b_{ij} \tilde{u}_{x_k x_j} \tilde{u}_{x_k x_i} - \sum_{i,j,k,l=1}^{n} 4 |\nabla \tilde{u}|^{-4} \, \tilde{u}_{x_k} \tilde{u}_{x_k x_j} b_{ij} \tilde{u}_{x_l} \tilde{u}_{x_l x_i} \right]. \tag{10}$$

We assume as we may that $\tilde{u}_{x_j} = 0$ for $j \neq 1$, since otherwise we rotate our coordinate system and use invariance of the p Laplace equation under rotations. Under this assumption we have

$$\begin{aligned} b_{11} &= (p-1)\,|\nabla \tilde{u}|^{p-2}, \\ b_{ii} &= |\nabla \tilde{u}|^{p-2} \quad i \neq 1, \\ b_{ij} &= 0 \quad i \neq j. \end{aligned}$$

Using these equalities in (10) we obtain, at x,

$$L\tau = 2|\nabla \tilde{u}|^{p-4}\left((p-1)\sum_{k=1}^{n}\tilde{u}^2_{x_k x_1} + \sum_{i=2,k=1}^{n}\tilde{u}^2_{x_k x_i} - 2(p-1)\tilde{u}^2_{x_1 x_1} - \sum_{i=2}^{n} 2u^2_{x_1 x_i}\right).$$

Collecting the $x_1 x_1$ and $x_1 x_i$ $(i \neq 1)$ derivatives yields

$$L\tau = 2|\nabla \tilde{u}|^{p-4}\left(-(p-1)\tilde{u}^2_{x_1 x_1} + (p-2)\sum_{k=2}^{n}\tilde{u}^2_{x_k x_1} + \sum_{k,i=2}^{n}\tilde{u}^2_{x_k x_i}\right). \tag{11}$$

The last sum contains the pure second derivatives of $\tilde{u}$ in the x_k direction when $k \neq 1$. These derivatives may be estimated using the p-Laplace equation for u at the point x, i.e., at x we have

$$(p-1)\tilde{u}_{x_1 x_1} + \sum_{k=2}^{n}\tilde{u}_{x_k x_k} = 0.$$

Solving for $\tilde{u}_{x_1 x_1}$, taking squares and using Hölder's inequality we see that

$$\sum_{k=2}^{n}\tilde{u}^2_{x_k x_k} \geq \frac{(p-1)^2}{n-1}\tilde{u}^2_{x_1 x_1}.$$

Substituting this expression into (11) gives

$$L\tau \geq 2|\nabla \tilde{u}|^{p-4}\left(\left(\tfrac{(p-1)^2}{n-1} - (p-1)\right)\tilde{u}^2_{x_1 x_1} + (p-2)\sum_{k=2}\tilde{u}^2_{x_k x_1} + \sum_{k,i=2,k\neq i}^{n}\tilde{u}^2_{x_k x_i}\right).$$

Thus, $L\tau \geq 0$ when $\frac{(p-1)^2}{n-1} - (p-1) = \frac{(p-1)(p-n)}{n-1} \geq 0$. In particular, $L\tau \geq 0$ if $p \geq n$. Note that when $p = n$ then $\tilde{u}(x) = \log|x|$ is n harmonic and $L(\log|\nabla \tilde{u}|) \equiv 0$ when $x \neq 0$.

3 Proof of Theorem 1.

Let $p, n, O, \hat{u}, \hat{\mu}, \rho, \hat{z}$, be as in Theorem 1 and suppose that λ is a positive nondecreasing function on $(0,1]$ with $\lim_{t\to 0} t^{1-n}\lambda(t) = 0$. Theorem 1 follows easily from the next proposition(See section 3.2).

3.1 Proof of Proposition 1

Proposition 1 *There exists $c = c(p, n)$ and a set $Q \subset \partial O \cap B(\hat{z}, \rho)$ with the following properties. $\hat{\mu}(\partial O \cap B(\hat{z}, \rho) \setminus Q) = 0$ and for every $w \in Q$ there are arbitrarily small $r = r(w), 0 < r \leq 10^{-10}$, such that*

$$(a)\ \bar{B}(w, 100r) \subset B(\hat{z}, \rho) \ \ \text{and} \ \ \hat{\mu}(B(w, 100r)) \leq c\, \hat{\mu}(B(w, r)).$$

Moreover there is a compact set $F = F(w, r) \subset \partial O \cap B(w, 20r)$ with

$$(b)\ H^{\lambda}(F) = 0 \ \ \text{and} \ \ \hat{\mu}(F) \geq \tfrac{1}{c}\, \hat{\mu}(B(w, 100r)).$$

Proof To prove (a) of Proposition 1 we note that $\hat{\mu}(B(x, t)) \neq 0$ whenever $x \in \partial O$ and $\partial O \cap B(x, t) \subset \partial O \cap B(\hat{z}, \rho)$ and $t > 0$ as follows from Lemma 3. Let

$$\Theta = \left\{ x \in \partial O \cap B(\hat{z}, \rho) : \liminf_{t \longrightarrow 0} \ \frac{\hat{\mu}(B(x, 100t))}{\hat{\mu}(B(x, t))} \geq c \right\}$$

If $x \in \Theta$, then there exists $t_0(x) > 0$ for which

$$\hat{\mu}(B(x, 100t)) \geq \frac{c}{2} \hat{\mu}(B(x, t)) \text{ for } 0 < t < t_0(x).$$

Iterating this inequality it follows that if c is large enough then

$$\lim_{t \longrightarrow 0} \frac{\hat{\mu}(B(x, t))}{t^{n+1}} = 0 \text{ whenever } x \in \Theta.$$

Since $H^{n+1}(\mathbb{R}^n) = 0$, we conclude that $\hat{\mu}(\Theta) = 0$. Thus we assume (a) holds for some $c' = c'(n), w \in \partial O \cap B(\hat{z}, \rho)$, and $r > 0$.

To prove (b) of Proposition 1 let

$$\gamma^{-1} = \max_{B(w, 10r)} \hat{u}$$

and put

$$u(x) = \gamma \hat{u}(w + rx) \text{ when } w + rx \in B(\hat{z}, \rho).$$

Let

$$\Omega = \{x : w + rx \in O \cap B(\hat{z}, \rho)\}.$$

Using translation and dilation invariance of the p Laplacian we find that u is p harmonic in Ω and if $\zeta = r^{-1}(\hat{z} - w)$, then u is continuous in $B(\zeta, \rho/r)$ with $u \equiv 0$ on $B(\zeta, \rho/r) \setminus \Omega$. Moreover there exists a measure μ on $\mathbb{R}^n$ with support in $\partial\Omega \cap \bar{B}(\zeta, \rho/r)$ corresponding to u. In fact if E is a Borel set and $T(E) = \{w + rx : x \in E\}$ then $\mu(E) = r^{p-n}\gamma^{p-1}\hat{\mu}(T(E))$. From Lemma 3 and Proposition 1 (a), we obtain for some $c = c(p, n) \geq 1$ and $2 \leq t \leq 50$ that

$$\frac{1}{c} \leq \mu(B(0, 1)) \leq \max_{B(0,2)} u \leq \max_{B(0,t)} u \leq c\, \mu(B(0, 100)) \leq c^2. \tag{12}$$

From (12) and the definition of u we observe that to prove Proposition 1 (b) it suffices to show that there exists a compact set $F' \subset B(0, 20)$ and $\hat{c} = \hat{c}(p, n) \geq 1$ with

$$\mu(F') \geq \frac{1}{\hat{c}} \ \ \text{and} \ \ H^{\lambda}(F') = 0. \tag{13}$$

To prove (13) we first show for given $\epsilon, \tau > 0$ that there exists a Borel set $E \subset B(0,20)$ and $c = c(p,n) \geq 1$ with

$$\phi_\tau^\lambda(E) \leq \epsilon \quad \text{and} \quad \mu(E) \geq \frac{1}{c}. \tag{14}$$

(13) follows easily from (14). Indeed, choose E_m relative to $\tau = \epsilon = 2^{-m}, m = 1, 2, \ldots$ and put

$$E = \bigcap_k \left(\bigcup_{m=k} E_m \right).$$

Then from measure theoretic arguments it follows that (13) is valid with F' replaced by E and $\hat{c}$ by c''. Using regularity of μ we then get (13) for a compact set $F' \subset E$. Thus to complete the proof of Proposition 1 we need only prove (14).

To prove (14) we note from the definition of u that $u(\tilde{z}) = 1$ for some $\tilde{z} \in \partial B(0,10)$. This note, (12), and Lemma 2 imply for some $c_- = c_-(p,n) \geq 1$ that

$$d(\tilde{z}, \partial\Omega) \geq \frac{1}{c_-}\,. \tag{15}$$

In fact otherwise it would follow from Lemma 2 that $\max_{B(0,20)} u$ is too large for (12) to hold.

Next let M be a large positive number and $0 < s < e^{-M}$. For the moment we allow M to vary but shall later fix it to satisfy several conditions. We then choose $s = s(M)$. First given $0 < \tilde{\tau} < \min(\tau, 10^{-5})$ choose M so large that if

$$z \in \partial\Omega \cap \bar{B}(0,15) \quad \text{and} \quad \mu(B(z,t)) = Mt^{n-1} \text{ for some } t = t(z) \leq 1, \text{ then } t \leq \tilde{\tau}. \tag{16}$$

Existence of $1 \leq M = M(\tilde{\tau})$ follows from (12). Next following Wolff [23] we observe from (16) that for each $z \in \partial\Omega \cap \bar{B}(0,15)$ there exists a largest $t = t(z)$, $s \leq t \leq \tilde{\tau}$, with either

$$\begin{array}{ll} (\alpha) & \mu(B(z,t)) = Mt^{n-1}, t > s, \\ \text{or} & \\ (\beta) & t = s. \end{array} \tag{17}$$

Using the Besicovitch covering theorem (see [20]) we now obtain a covering $\{B(z_j,t_j)\}_1^N$ of $\partial\Omega \cap \bar{B}(0,15)$, where $t_j = t(z_j)$ is the maximal t for which either (17) (α) or (β) holds. Moreover each point of $\bigcup_{j=1}^N B(z_j,t_j)$ lies in at most $c = c(n)$ of $\{B(z_j,t_j)\}_1^N$. Let $c_-, \tilde{z}$, be as in (15) and set $r_1 = (8c_-)^{-1}$. Choosing $\tilde{\tau}$ smaller (so M larger) if necessary we may assume, thanks to (16), that

$$\bigcup_{j=1}^N \bar{B}(z_j, 6t_j) \cap B(\tilde{z}, 6r_1) = \emptyset. \tag{18}$$

Also put

$$\Omega' = \Omega \cap B(0,15) \setminus \bigcup_{j=1}^N \bar{B}(z_j,t_j)$$

and

$$D = \Omega' \setminus \bar{B}(\tilde{z}, 2r_1).$$

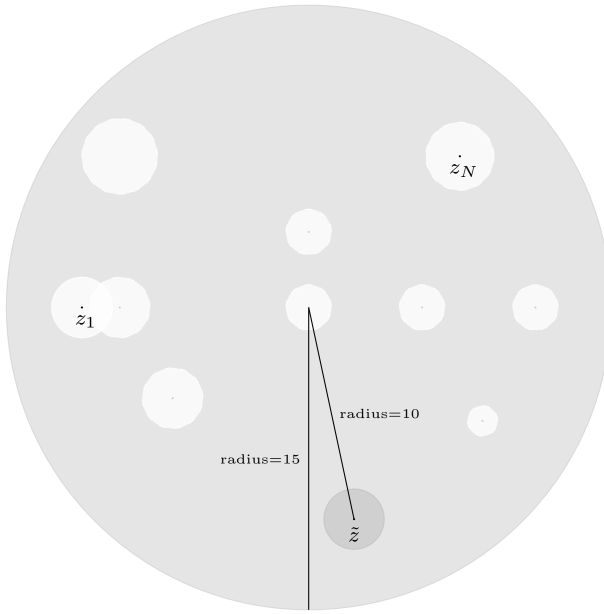

Fig. 1: An example of $\Omega' = \Omega \cap B(0,15) \setminus \bigcup_{j=1}^{N} \bar{B}(z_j, t_j)$.

Let u' be the p harmonic function in D with continuous boundary values,

$$u'(x) \equiv \begin{cases} 0 & \text{when } x \in \partial\Omega' \\ \min\limits_{\bar{B}(\tilde{z},2r_1)} u & \text{when } x \in \partial B(\tilde{z}, 2r_1). \end{cases}$$

Extend u' continuously to $\bar{B}(0,15)$ (also denoted u') by putting

$$u'(x) \equiv \begin{cases} 0 & \text{when } x \in \bar{B}(0,15) \setminus \Omega' \\ \min\limits_{\bar{B}(\tilde{z},2r_1)} u & \text{when } x \in \bar{B}(\tilde{z}, 2r_1). \end{cases}$$

We note that $u' \le u$ on ∂D so by the maximum principle for p harmonic functions $u' \le u$ in D. Also, ∂D is locally (n, r_0') uniformly fat where r_0' depends only on p, n, and r_0 in Theorem 1.

To continue the proof of (14) we shall need several lemmas.

Lemma 7 *If $x \in D$, then*

$$|\nabla u'(x)| \le c\, M^{\frac{1}{p-1}}\,.$$

Proof To prove Lemma 7 let $x \in D$ and choose $y \in \partial D$ with $|x-y| = d(x, \partial D) = d$. If $y \in \partial B(z_k, t_k)$ and $x \in B(z_k, 2t_k)$ we put

$$f(w) = A\left(|w - z_k|^{\frac{p-n}{p-1}} - t_k^{\frac{p-n}{p-1}}\right), w \in B(z_k, 2t_k) \setminus \bar{B}(z_k, t_k),$$

when $p > n$ and

$$f(w) = A\left(\log|w - z_k| - \log t_k\right), w \in B(z_k, 2t_k) \setminus \bar{B}(z_k, t_k)$$

when $p = n$. Then $f \equiv 0$ on $\partial B(z_k, t_k)$ and A is chosen so that

$$f \equiv \max_{B(z_k, 2t_k)} u \text{ on } \partial B(z_k, 2t_k).$$

Then from $u' \leq u$ and the maximum principle for p harmonic functions, $u' \leq f$ in $B(z_k, 2t_k) \setminus \bar{B}(z_k, t_k)$. Using this inequality and applying Lemma 4 to u' we conclude that

$$|\nabla u'(x)| \leq \frac{c}{d}\, u'(x) \leq \frac{c}{d}\, f(x) \leq \frac{c^2}{t_k} \max_{B(z_k, 2t_k)} u. \tag{19}$$

Also from Lemma 3 and (16)-(18) we find that

$$t_k^{1-p} \max_{B(z_k, 2t_k)} u^{p-1} \leq c\, t_k^{1-n}\, \mu(B(z_k, 4t_k)) \leq c^2 M. \tag{20}$$

Taking $1/(p-1)$ powers of both sides of (20) and using the resulting inequality in (19) we get Lemma 7 when $y \in \partial B(z_k, t_k)$ and $x \in D \cap B(z_k, 2t_k)$. If $y \in \partial B(0, 15)$ or $\partial B(\tilde{z}, 2r_1)$ a similar argument applies. Thus there is an open neighborhood, say W, containing ∂D for which the conclusion of Lemma 7 is valid when $x \in W \cap D$. From this conclusion, Lemma 6 applied to u', and a maximum principle for weak sub solutions to L, we conclude that Lemma 7 is valid in D. $\square$

Next we prove

Lemma 8 *The functions $|\nabla u'|^{p-2}\, |u'_{x_k x_i}|$ for $1 \leq i, k \leq n$ are all integrable on D*

$$\sum_{i,k=1}^{n} \int_D |\nabla u'|^{p-2}\, |u'_{x_k x_i}|\, dx < \infty$$

Proof Let $\Lambda \subset \partial \Omega'$ be the set of points where $\partial\Omega'$ is not smooth. Clearly $H^{n-1}(\Lambda) = 0$. If $\hat{x} \in \partial D \setminus \Lambda$, then $\hat{x}$ lies in exactly one of the finite number of spheres which contain points of ∂D. Let $d'(\hat{x})$ denote the distance from $\hat{x}$ to the union of spheres not containing $\hat{x}$ but containing points of ∂D. If $d' = d'(\hat{x}) < s/100$, then from Lemma 5 applied to u' we see that each component of $\nabla u'$ has a Hölder continuous extension to $B(\hat{x}, 3d'/4)$. Also from Hölder, Lemma 5, and Lemma 7 we see that

$$\begin{aligned}
\frac{1}{c} \sum_{i,k=1}^{n} \int_{D \cap B(\hat{x}, \frac{d'}{8})} |\nabla u'|^{p-2}\, |u'_{x_i x_k}|\, dx &\leq (d')^{\frac{n}{2}} M^{\frac{p-2}{2(p-1)}} \sum_{i,k=1}^{n} \left(\int_{D \cap B(\hat{x}, \frac{d'}{8})} |\nabla u'|^{p-2}\, |u'_{x_i x_k}|^2\, dx \right)^{\frac{1}{2}} \\
&\leq c (d')^{\frac{(n-2)}{2}} M^{\frac{p-2}{2(p-1)}} \left(\int_{D \cap B(\hat{x}, \frac{d'}{2})} |\nabla u'|^p\, dx \right)^{\frac{1}{2}} \\
&\leq c^2 M\, (d')^{(n-1)}.
\end{aligned} \tag{21}$$

To prove Lemma 8 we assume as we may that $B(z_l, t_l) \not\subset B(z_\nu, t_\nu)$ when $\nu \neq l$, since otherwise we discard one of these balls. Also from a well known covering theorem we get a covering $\{B(y_j, \frac{1}{20} d'(y_j))\}$ of $\partial D \setminus \Lambda$ with $\{B(y_j, \frac{1}{100} d'(y_j))\}$, pairwise disjoint. From (21) we find that

$$\begin{aligned}\sum_{i,j,k} \int\limits_{D \cap B(y_j, \frac{1}{8} d'(y_j))} |\nabla u'|^{p-2} |u'_{x_k x_i}| dx &\leq c\, M \sum_j (d'(y_j))^{n-1} \\ &\leq c^2 M H^{n-1}(\partial D).\end{aligned} \tag{22}$$

For short we now write $d(x)$ for $d(x, \partial D)$ and choose a covering $\{B(x_m, \frac{1}{2} d(x_m)\}$ of D with $\{B(x_m, \frac{1}{20} d(x_m)\}$, pairwise disjoint. We note that if $x \in D$ and $y \in \partial D$ with $|y - x| = d(x)$, then $y \in \partial D \setminus \Lambda$. Indeed otherwise y would be on the boundary of at least two balls contained in the complement of D and so by the no containment assumption above, would have to intersect $B(x, d(x))$, which clearly is a contradiction. Also we note that if $d(x) \leq 1000s$, then $d(x) \leq \kappa\, d'(y)$ where κ can depend on various quantities including the configuration of the $B(z_k, t_k)$ balls but is independent of $x \in D$ with $d(x) \leq 1000s$. Indeed from the no containment assumption one just needs to consider $d(x)/d'(y)$ as $d(x), d'(y) \to 0$. To do this suppose $z \in \Lambda$ with $|y - z| = d'(y)$. Then one sees, from consideration of half planes containing z and tangent to two intersecting spheres, that x, y eventually lie in a truncated cone of height γ with vertex at z, and of angle opening $\leq \alpha < \pi/2$, where α, γ are independent of x, y, z. Moreover the complement of this truncated cone in a certain hemisphere of radius γ with center z lies outside of Ω'. Then a ballpark estimate using trigonometry gives $d'(y) \geq (1 - \sin\alpha) d(x)$(See Figure 2).

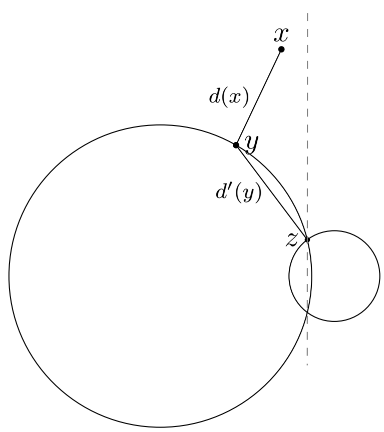

Fig. 2: $d'(y) \geq (1 - \sin\alpha) d(x)$.

From this analysis and our choice of covering of D we see that for a given $B(x_m, \frac{1}{2} d(x_m))$ with $d(x_m) < 1000s$, there exists $j = j(m)$ with $B(x_m, \frac{1}{2} d(x_m)) \subset B(y_j, \kappa' d'(y_j))$ for some $0 < \kappa' < \infty$ independent of m.

Let $S_l, l = 1, 2, 3$, be disjoint sets of integers defined as follows.

$$\begin{cases} m \in S_1 \text{ if } \ d(x_m) \geq 1000s, \\ m \in S_2 \text{ if } \ m \notin S_1 \text{ and there does not exist } j \text{ with } B(x_m, \frac{1}{2}d(x_m)) \subset B(y_j, \frac{1}{8}d'(y_j)), \\ m \in S_3 \text{ if } \ m \text{ not in either } S_1 \text{ or } S_2. \end{cases}$$

Let

$$K_l = \sum_{m \in S_l} \int\limits_{D \cap B(x_m, \frac{1}{2}d(x_m))} |\nabla u'|^{p-2} |u'_{x_k x_i}| dx \text{ for } l = 1, 2, 3.$$

Then

$$\int\limits_D |\nabla u'|^{p-2} |u'_{x_k x_i}| dx \ \leq \ K_1 \ + \ K_2 \ + \ K_3. \tag{23}$$

From Lemma 4 and the same argument as in (21) we see that

$$K_1 \leq c\, M \sum_{m \in S_1} d(x_m)^{n-1} \leq c^2 M s^{-1} \tag{24}$$

where we have used disjointness of our covering, $\{B(x_m, \frac{1}{20}d(x_m))\}$. Using disjointness of these balls and (22) we get

$$K_3 \leq c\, M H^{n-1}(\partial D). \tag{25}$$

Finally if $m \in S_2$, then as discussed earlier there exists $j = j(m)$ with $d(x_m) \approx d'(y_j)$, where proportionality constants are independent of m, so $B(x_m, \frac{1}{2}d(x_m)) \subset B(y_j, \kappa' d'(y_j))$. From disjointness of $\{B(x_m, \frac{1}{20}d(x_m))\}$ and a volume type argument we deduce that each j corresponds to at most κ'' integers $m \in S_3$ where κ'' is independent of j. From this fact, (21), and disjointness of $\{B(y_j, \frac{1}{100}d'(y_j))\}$ we conclude that there is a $\tilde{\kappa}, 0 < \tilde{\kappa} < \infty$, with

$$K_2 \ \leq \ \tilde{\kappa} M \sum_{m \in S_2} d(x_m)^{n-1} \leq \tilde{\kappa}^2 M \sum_j d'(y_j)^{n-1} \ \leq \tilde{\kappa}^3 M H^{n-1}(\partial D). \tag{26}$$

Using (24)-(26) in (23) we find that Lemma 8 is valid. □

Recall that $\nabla u'$ is Hölder continuous in $\bar{D} \setminus \Lambda$. We use this recollection and Lemmas 7, 8, to prove

Lemma 9 *There exists $c = c(p, n)$ such that*

$$\int\limits_{\partial D} |\nabla u'|^{p-1} |\log |\nabla u'||\, dH^{n-1} \ \leq \ c \log M.$$

Proof From smoothness of u' in $\bar{D} \setminus \Lambda$, (2), and integration by parts, we see that

$$d\mu'/dH^{n-1} = |\nabla u'|^{p-1} > 0 \text{ on } \partial\Omega' \setminus \Lambda. \tag{27}$$

We claim for some $c = c(p, n) \geq 1$ that

$$\frac{1}{c} \leq \mu'(\partial\Omega' \cap B(0, 10)) \leq \mu'(\partial\Omega') \leq c. \tag{28}$$

To prove the left hand inequality in (28) we first observe from $u(\tilde{z}) = 1$ and Lemmas 1, 2, and (18) that $c^* u' \geq 1$ on $\partial B(\tilde{z}, 4r_1)$ for some $c^* = c^*(p, n) \geq 1$. Let l denote the line from the origin through $\tilde z$ and let ζ_1 be the point on this line segment in $\partial B(\tilde z, 4r_1) \cap B(0,10)$. Let ζ_2 be the point on the line segment from ζ_1 to the origin with $d(\zeta_2, \partial\Omega') = \frac{1}{20} r_1$ while $d(\zeta, \partial\Omega') > \frac{1}{20} r_1$ at every other point on the line segment from ζ_1 to ζ_2. Then from (15), Lemma 1, and the above discussion we see that $c^{**} u(\zeta_2) \geq 1$ for some $c^{**}(p,n) \geq 1$. Also, $B(\zeta_2, \frac{1}{2} r_1) \subset B(0,10)$. Let $\hat\zeta$ be the point in $\partial\Omega'$ with $|\hat\zeta - \zeta_2| = d(\zeta_2, \partial\Omega')$. Applying Lemma 3 with $w = \hat\zeta, r = 2d(\zeta_2, \partial\Omega')$, we deduce that the left hand inequality in (28) is valid. The right hand inequality in this claim follows once again from Lemma 3 and $u' \leq u$.

Let

$$\log^+ t = \max\{\log t, 0\}$$

and

$$\log^- t = \log^+(1/t)$$

for $t \in (0,\infty)$. From Lemma 7, (27), (28), and $H^{n-1}(\Lambda) = 0$ we obtain for some $c = c(p,n) \geq 1$,

$$\int\limits_{\partial\Omega'} |\nabla u'|^{p-1} \log^+ |\nabla u'| \, dH^{n-1} \leq c \log M \; \mu'(\partial\Omega') \leq c^2 \log M. \tag{29}$$

To estimate $\log^- |\nabla u'|$, fix $\eta, -\infty \leq \eta \leq -1$, and let $v'(x) = \max\{\log|\nabla u'|, \eta\}$ when $x \in \bar D \setminus \Lambda$. Given a small $\theta > 0$ let

$$\Lambda(\theta) = \{x \in D : d(x,\Lambda) \leq \theta\} \text{ and } D(\theta) = D \setminus \Lambda(\theta).$$

From Lemma 4 and Lemmas 7, 8 we deduce that $|\nabla u'|^{p-2} u'_{x_i}$ has a $W^{1,2}(D(\theta))$ extension with distributional derivative $(|\nabla u'|^{p-2} u'_{x_i})_{x_j} = 0$ when $|\nabla u'| = 0$ and $1 \leq i,j \leq n$. Moreover these functions are continuous near $\partial D(\theta)$ thanks to Lemmas 4 and 5. Let $\{b_{ik}\}, L$, be as defined in (3), (4) relative to u' and note from the above discussion that

$$Lu'(x) = (p-1)\nabla \cdot \left(|\nabla u'|^{p-2} \nabla u'\right)(x) = 0$$

exists pointwise for almost every $x \in D(\theta)$. Put

$$I(\theta) = \int\limits_{D(\theta)} Lu' \, v' \, dx + \int\limits_{D(\theta)} \sum_{i,k=1}^{n} b_{ik} u'_{x_k} \, v'_{x_i} dx = I_1(\theta) + I_2(\theta). \tag{30}$$

Clearly $I_1(\theta) = 0$. To handle $I_2(\theta)$ we first argue as in (19), i.e, use a barrier argument, and second use Lemma 5 to deduce for some $c = c(p,n) \geq 1$, that if $r_2 = (1 + c^{-1}) r_1$, then

$$\frac{1}{c} \leq |\nabla u'| \leq c \text{ on } \bar B(\tilde z, 2r_2) \setminus B(\tilde z, 2r_1). \tag{31}$$

Let ψ be infinitely differentiable and $0 \leq \psi \leq 1$ on $\mathbb{R}^n$ with $\psi \equiv 1$ on $\mathbb{R}^n \setminus B(\tilde z, 2r_2)$ and $|\nabla\psi| \leq c r_1^{-1} \leq c^2$, where the last inequality follows from (15) and the definition of r_1. Suppose also that ψ vanishes in an open set containing $\bar B(\tilde z, 2r_1)$. Then

$$\begin{aligned} I_2(\theta) &= \int\limits_{D(\theta)} \sum_{i,k=1}^{n} b_{ik} (\psi u')_{x_k} \, v'_{x_i} dx + \int\limits_{D(\theta)} \sum_{i,k=1}^{n} b_{ik} ((1-\psi) u')_{x_k} \, v'_{x_i} dx \\ &= I_{21}(\theta) + I_{22}(\theta). \end{aligned} \tag{32}$$

From Lemmas 4, 5, (31), and an argument similar to the one in (21) we deduce for some $c = c(p, n) \geq 1$ that
$$|I_{22}| \leq c. \tag{33}$$

Turning to $I_{21}(\theta)$ we note from Lemmas 7 and 8 that the integrand in the integral defining $I_{21}(\theta)$ is dominated by an integrable function independent of θ. Thus from the Lebesgue dominated convergence theorem,
$$\lim_{\theta\to 0} I_{21}(\theta) = \int_D \sum_{i,k=1}^n b_{ik}(\psi u')_{x_k}\, v'_{x_i} dx = I'. \tag{34}$$
We assert that
$$I' \leq 0. \tag{35}$$
To verify this assertion let $u'' = u''(\delta) = \max(u' - \delta, 0)$. Using the convolution of $\psi u''$ with an approximate identity and taking limits we see from Lemma 6 that
$$\int_D \sum_{i,k=1}^n b_{ik}(\psi u'')_{x_k}\, v'_{x_i} dx \leq 0\,.$$
Now again from Lemmas 7 and 8, we observe that the above integrand is dominated by an integrable function independent of δ. Using this fact, the above inequality, and the Lebesgue dominated convergence theorem we get assertion (35). Using (30) - (35) we conclude (since $I_{22}(\theta)$ is independent of θ) that
$$\lim_{\theta\to 0} I(\theta) \leq c. \tag{36}$$
On the other hand from [7, Chapter 5] and the discussion above (30) we see that integration by parts can be used to get
$$I_1(\theta) = -I_2(\theta) + \int_{\partial D(\theta)} v' \sum_{i,k=1}^n b_{ik}\, u'_{x_k} \nu_i dH^{n-1} \tag{37}$$
where $\nu = (\nu_1, \dots \nu_n)$ is the outer unit normal to $\partial D(\theta)$. From (31) we see that
$$\left| \int_{\partial B(\tilde z, 2r_1)} v' \sum_{i,k=1}^n b_{ik}\, u'_{x_k} \nu_i dH^{n-1} \right| \leq c = c(p, n). \tag{38}$$
From Lemma 7, dominated convergence, and the definition of $D(\theta)$, we have
$$\int_{\partial D(\theta)\setminus \partial B(\tilde z, 2r_1)} v' \sum_{i,k=1}^n b_{ik}\, u'_{x_k} \nu_i dH^{n-1} \to \int_{\partial \Omega' \setminus \Lambda} v' \sum_{i,k=1}^n b_{ik}\, u'_{x_k} \nu_i dH^{n-1} \text{ as } \theta \to 0. \tag{39}$$

Observe that $\nu = -\frac{\nabla u'}{|\nabla u'|}$ on $\partial\Omega' \setminus \Lambda$. From this observation and (4) we calculate
$$\begin{aligned}\sum_{i,k=1}^n b_{ik}\, u'_{x_k} \nu_i &= -\sum_{i,k=1}^n |\nabla u'|^{p-5}[(p-2)(u')^2_{x_i}(u')^2_{x_k} + \delta_{ik}|\nabla u'|^2] u_{x_i}\, u_{x_k} \\ &= -(p-1)|\nabla u'|^{p-1}.\end{aligned} \tag{40}$$

From (30), (36)-(40) we find that

$$- (p-1) \int\limits_{\partial\Omega'} v \, |\nabla u'|^{p-1} \, dH^{n-1} \leq \lim_{\theta\to 0} I(\theta) + c \, \leq \, 2c. \tag{41}$$

Letting $\eta \to -\infty$ in (41) and using the monotone convergence theorem we see that (41) holds with v replaced by $\log |\nabla u|$. Finally from (41) for $\log |\nabla u|$ and (29) we conclude the validity of Lemma 9. $\square$

With these lemmas in hand, we go back to the proof of (14) and Proposition 1b. We note from Lemma 3 and $u' \leq u$ that for given $j, 1 \leq j \leq N$,

$$t_j^{1-n} \, \mu'(\bar{B}(z_j, t_j)) \leq c \, t_j^{1-p} \max_{B(z_j, 2t_j)} u^{p-1} \leq c^2 \, t_j^{1-n} \, \mu(B(z_j, 4t_j)). \tag{42}$$

For given $A >> 1$, we see from (17) that $\{1, 2, \ldots, N\}$ can be divided into disjoint subsets Φ_1, Φ_2, Φ_3, as follows.

$$\begin{cases} j \in \Phi_1 \text{ if } t_j > s, \\ j \in \Phi_2 \text{ if } t_j = s \text{ and } |\nabla u'|^{p-1}(x) \geq M^{-A}, \text{ for some } x \in \partial\Omega' \cap \partial B(z_j, t_j) \setminus \Lambda \\ j \in \Phi_3 \text{ if } j \text{ is not in } \Phi_1 \text{ or } \Phi_2. \end{cases}$$

Let $t_j' = t_j$ when $j \in \Phi_1$ and $t_j' = 4s$ when $j \in \Phi_2$. To prove (14) set

$$E = \partial\Omega \cap \bigcup_{j \in \Phi_1 \cup \Phi_2} B(z_j, t_j').$$

To estimate $\phi_\tau^\lambda(E)$ we first observe that if

$$x \in \bigcup_{j \in \Phi_1 \cup \Phi_2} B(z_j, t_j') \text{ then } x \text{ lies in at most } c = c(n) \text{ of } \{B(z_j, t_j')\}. \tag{43}$$

This observation can be proved using $t_j \geq s, 1 \leq j \leq N$, a volume type argument, and the fact that $\{B(z_j, t_j)\}_1^N$ is a Besicovitch covering of $\partial\Omega \cap \bar{B}(0, 15)$. If $j \in \Phi_2$ we get from (19), (42), that for some $c = c(p, n) \geq 1$

$$M^{-A} \leq |\nabla u'(x)|^{p-1} \leq c \, s^{1-n} \mu(B(z_j, 4s)) \, .$$

Rearranging this inequality, summing, and using (12), (43), we see that

$$\sum_{j \in \Phi_2} (t_j')^{n-1} \, \leq \, \tilde{c} \, M^A \mu(\bigcup_{j \in \Phi_2} B(z_j, t_j')) \leq (\tilde{c})^2 \, M^A$$

provided $\tilde{c} = \tilde{c}(p, n)$ is large enough. Now since $t_j' = s$ for all $j \in \Phi_2$ we may for given A, M, ϵ choose $s > 0$ so small that

$$s^{1-n} \lambda(s) \leq \frac{\epsilon}{2(\tilde{c})^2 M^A} \tag{44}$$

where we have used the definition of λ. Using this choice of s in the above display we get

$$\sum_{j \in \Phi_2} \lambda(t_j') \, \leq \, \epsilon/2. \tag{45}$$

On the other hand we may suppose $\bar{\tau}$ in (16) is so small that $\lambda(t_j) \le t_j^{n-1}$ for $1 \le j \le N$. Then from (12), (17), and (43), we see that

$$\begin{aligned} \sum_{j\in\Phi_1} \lambda(t'_j) &\le \sum_{j\in\Phi_1} (t'_j)^{n-1} \\ &= M^{-1} \sum_{j\in\Phi_1} \mu(B(z_j,t_j)) \le \epsilon/2 \end{aligned} \tag{46}$$

provided $M = M(\epsilon)$ is chosen large enough. Fix M satisfying all of the above requirements. In view of (45), (46), we have proved the left hand inequality in (14) for E as defined above, i.e. $\phi_\tau^\lambda(E) \le \epsilon$.

To prove the right hand inequality in (14) we use Lemma 9 and the definition of Φ_3 to obtain

$$\begin{aligned} \mu'\left(\partial\Omega' \cap \bigcup_{j\in\Phi_3} \bar{B}(z_j,t_j)\right) &\le \mu'\left(\{x\in\partial\Omega' : |\nabla u'(x)|^{p-1} \le M^{-A}\}\right) \\ &\le (p-1)(A\log M)^{-1} \int_{\partial\Omega'} |\nabla u'|^{p-1}\,|\log|\nabla u'||dH^{n-1} \\ &\le \frac{c}{A}. \end{aligned} \tag{47}$$

Choosing $A = A(n)$ large enough we have from (28), (47),

$$\mu'\left(\bigcup_{j\in\Phi_1\cup\Phi_2} B(0,10)\cap\bar{B}(z_j,t_j)\right) \ge \mu'(B(0,10)) - \mu'\left(\bigcup_{j\in\Phi_3}\bar{B}(z_j,t_j)\right) \ge c_*^{-1} \tag{48}$$

for some $c_*(p,n)$. Finally from (42), (43), and (48), we get for some $c = c(p,n) \ge 1$ that

$$\mu(E) \ge c^{-1} \sum_{j\in\Phi_1\cup\Phi_2} \mu(\bar{B}(z_j,t'_j)) \ge c^{-2} \sum_{j\in\Phi_1\cup\Phi_2} \mu'(\bar{B}(z_j,t_j)) \ge c^{-3}. \tag{49}$$

For $j \in \Phi_1$ we have used the definition of t_j so that

$$\mu(B(z_j,4t_j)) < M4^{n-1}t_j^{n-1} = 4^{n-1}\mu(B(z_j,t_j)) = 4^{n-1}\mu(B(z_j,t'_j))$$

Thus (14) is valid. Proposition 1 follows from (14) and our earlier remarks. □

3.2 Proof of Theorem 1

Next we show for λ, Q as in Proposition 1 that there exists a Borel set Q_1 with

$$Q_1 \subset Q,\ \hat{\mu}(\partial O \cap B(\hat{z},\rho)\setminus Q_1) = 0, \text{ and } H^\lambda(Q_1) = 0. \tag{50}$$

To prove (50) we assume, as we may, that $\hat{\mu}(\partial O \cap B(\hat{z},\rho)) < \infty$ since otherwise we can write $\partial O \cap B(\hat{z},\rho)$ as a countable union of Borel sets with finite $\hat{\mu}$ measure and apply the following argument in each set. Under this assumption we can use Proposition 1 and a Vitali type covering argument (see [20]), as well as induction to get compact sets $\{F_l\}$, $F_l \subset Q$, with $F_k \cap F_j = \emptyset$, $k \neq j$, $\hat\mu(F_1) > 0$ and with

$$c'\hat\mu(F_{m+1}) \geq \hat\mu(Q \setminus \bigcup_{l=1}^{m} F_l), m = 1, 2, \dots,$$

for some $c' = c'(p,n) \geq 1$. Moreover $H^\lambda(F_l) = 0$ for all l. Then $Q_1 = \bigcup_{l=1}^\infty F_l$ has the desired properties as follows from measure theoretic arguments.

To prove Theorem 1 we first note from a covering argument as in [15] or [23] that if

$$P = \{x \in \partial O \cap B(\hat z, \rho) : \limsup_{t\to 0} \frac{\hat\mu(B(x,t))}{t^{n-1}} > 0\},$$

then P has σ finite H^{n-1} measure. For completeness we prove this statement after finishing the proof of Theorem 1. Thus to prove Theorem 1 it suffices to show that

$$\hat\mu(Q_1 \setminus P) = 0. \tag{51}$$

Indeed otherwise from Egoroff's theorem there exists a compact set $K \subset Q_1 \setminus P$ with

$$\hat\mu(K) > 0 \text{ and } \lim_{t\to 0} \frac{\hat\mu(B(x,t))}{t^{n-1}} = 0 \text{ uniformly for } x \in K. \tag{52}$$

Choose $\alpha_k \in (0,1), k = 1, 2, \dots,$ with $\alpha_{k+1} < \alpha_k/2$ and so that

$$\sup_{0<t\leq\alpha_k} \frac{\hat\mu(B(x,t))}{t^{n-1}} \leq 2^{-2k} \text{ for all } x \in K.$$

Let $\alpha_0 = 1$. With $(\alpha_k)_0^\infty$ now chosen, define $\lambda(t)$ on $(0,1]$ by $\lambda(\alpha_k) = 2^{-k}(\alpha_k)^{n-1}, k = 0, 1, \dots,$ and $t^{1-n}\lambda(t)$ is linear for t in the intervals $[\alpha_{k+1}, \alpha_k]$ for $k = 0, 1, \dots$ Put $\lambda(0) = 0$. Clearly $t^{1-n}\lambda(t) \to 0$ as $t \to 0$. Also, if $\alpha_{k+1} \leq t \leq \alpha_k$, and $x \in K$, then

$$\frac{\hat\mu(B(x,t)}{\lambda(t)} \leq 2^{1-k}. \tag{53}$$

Given m a positive integer we note from (50) that there is a covering $\{B(x_j, r_j)\}$ of K with $r_j \leq \alpha_m/2$ for all j and

$$\sum_j \lambda(2r_j) \;\leq\; 1$$

We may assume that there is an $x_j' \in K \cap B(x_j, r_j)$ for each j since otherwise we discard $B(x_j, r_j)$. Moreover from (53) we see that

$$\hat\mu(K) \leq \sum_j \hat\mu(B(x_j', 2r_j)) \leq 2^{1-m} \sum_j \lambda(2r_j) \leq 2^{1-m}.$$

Since m is arbitrary we have reached a contradiction to $\hat\mu(K) > 0$ in (52). From this contradiction we conclude first (51) and second Theorem 1. $\square$

To prove that P has σ finite H^{n-1} measure we once again may assume $\hat\mu(\partial O \cap B(\hat z, \hat\rho)) < \infty$. Let

$$P_m = \{x \in P : \limsup_{t\to 0} t^{1-n}\hat\mu(B(x,t)) > \frac{1}{m}\}$$

for $m = 1, 2, \ldots$ Given $\delta > 0$ we choose a Besicovitch covering $\{B(y_i, r_i)\}$ of P_m with $y_i \in P_m$, $r_i \leq \delta, B(y_i, r_i) \subset B(\hat{z}, \rho)$ and

$$\mu(B(y_i, r_i)) > \frac{r_i^{n-1}}{m}.$$

Thus

$$\sum_i r_i^{n-1} < m \sum_i \hat{\mu}(B(x_i, r_i)) \leq c\, m\, \hat{\mu}(\partial O \cap B(\hat{z}, \rho)) < \infty. \tag{54}$$

Letting $\delta \to 0$ and using the definition of H^{n-1} measure we conclude from (54) that $H^{n-1}(P_m) < \infty$. Hence P has σ finite H^{n-1} measure.

4 Closing Remarks

The existence of a measure, say μ, corresponding to a positive weak solution u in $O \cap B(\hat{z}, r)$ with vanishing boundary values, as in (2), can be shown for a large class of divergence form partial differential equations. What can be said about H-dim μ? What can be said about analogues of Theorems 1, 2? Regarding these questions we note that Akman in [1] has considered PDE's whose Euler equations arise from minimization problems with integrands involving $f(\nabla v)$ and $v \in W^{1,p}$. More specifically for fixed $p, 1 < p < \infty$, the function $f : \mathbb{R}^2 \setminus \{0\} \to (0, \infty)$, is homogeneous of degree p on $\mathbb{R}^2$. That is,

$$f(\eta) = |\eta|^p f\left(\frac{\eta}{|\eta|}\right) > 0 \text{ when } \eta = (\eta_1, \eta_2) \in \mathbb{R}^2 \setminus \{0\}.$$

Also $\nabla f = (f_{\eta_1}, f_{\eta_2})$ is δ monotone on $\mathbb{R}^2$ for some $\delta > 0$ (see [3] for a definition of δ monotone). In [1], Akman considers weak solutions to the Euler-Lagrange equation,

$$\sum_{k=1}^{2} \frac{\partial}{\partial x_k} \left(\frac{\partial f}{\partial \eta_k}(\nabla u(x)) \right) = 0 \text{ when } x = (x_1, x_2) \in \Omega \cap N, \tag{55}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded simply connected domain and N is a neighborhood of $\partial\Omega$. Assume also that $u > 0$ is continuous in N with $u \equiv 0$ in $N \setminus \Omega$. Under these assumptions it follows that there exists a unique finite positive Borel measure μ with support in $\partial\Omega$ satisfying

$$\int_{\mathbb{R}^2} \langle \nabla f(\nabla u), \nabla \phi \rangle dA = -\int_{\partial\Omega} \phi d\mu$$

whenever $\phi \in C_0^\infty(N)$. He proves

Theorem 4 *Let p, f, Ω, N, u, μ be as above and put*

$$\lambda(r) = r \exp\left[A \sqrt{\log \frac{1}{r} \log\log \frac{1}{r}} \right] \text{ for } 0 < r < 10^{-6}.$$

(a) *If $p \geq 2$, there exists $A = A(p) \leq -1$ such that μ is concentrated on a set of $\sigma-$finite H^λ Hausdorff measure.*

(b) *If $1 < p \leq 2$, there exists $A = A(p) \geq 1$, such that μ is absolutely continuous with respect to H^λ Hausdorff measure.*

For $p = 2$ and $f(\eta) = |\eta|^p$ the above theorem is slightly weaker than Theorem 2. It is easily seen that Theorem 4 implies

$$\text{H-dim } \mu \leq 1 \text{ for } p \geq 2 \text{ and H-dim } \mu \geq 1 \text{ for } 1 < p \leq 2.$$

A key argument in the proof of Theorem 4 involves showing that $\zeta = \log f(\nabla u)$ is a weak subsolution, supersolution or solution to

$$L\zeta(x) = \sum_{k,j=1}^{2} \frac{\partial}{\partial x_k} \left(f_{\eta_k \eta_j}(\nabla u(z)) \frac{\partial \zeta(x)}{\partial x_j} \right) \text{ when } x \in \Omega \cap N$$

and $p > 2, 1 < p < 2$, $p = 2$, respectively. In [2] this was shown pointwise at $x \in \Omega \cap N$ when $\nabla u, f,$ are sufficiently smooth and $\nabla u(x) \neq 0$. We plan to use this fact and the technique in Theorem 4 to prove analogues of Theorem 4 when $n = 2$ and also higher dimensional analogues. The case $p = n$ in Theorem 1 and $p = 2$ in the proposed generalization of Theorem 4 are particularly interesting. Can one for example do away with the uniform fatness assumption in Theorem 1 or the proposed generalization of Theorem 4 when $p = 2, n = 2$? The argument in [23] and [10] relies on a certain integral inequality (see Lemma 3.1 in [10]).

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